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Tuesday, 20th October 2020

Dear Panel Members

REVIEW by scientific cosupervisor, Professor H. Dugald Macpherson (University of Leeds, UK) on the dissertation of Sayan Samatovich Baizhanov on the theme 'Properties preservation of expansion of models of NIP theories' submitted for the degree of Doctor of Philosophy (PhD) in the specialty 6D060100 – Mathematics

This PhD thesis is in model theory, a thriving branch of mathematical logic in which key concepts and tools of logic (languages, structures, expressibility, definability) are focussed on structures of mathematics such as graphs, groups, fields, orders, and enrichments of these. Key notions are those of a first order structure M (such as an ordered field), and the *theory* $\mathrm{Th}(M)$ of M (the set of first order sentences which are true of M). A central theme is to divide first order theories into *tame* ones (where for example we understand the *definable sets*, that is, solution sets of first order formulas, and in some cases we classify the model of the theory), and *wild* ones (where for example definable sets are out of reach). Historically, the first approximation of tame theories was Shelah's notion of *stable* theory, but more recently the notion of tameness has broadened, and would now normally include *NIP* theories (those which do not have the independence property). The NIP theories include the immensely important *o-minimal* theories, and also the wider class of *weakly o-minimal* theories. Models of the latter carry a definable total order (and possibly additional structure) and have the property that every definable subset of the universe is a finite union of convex sets.

Weak o-minimality is at the heart of Sayan Baizhanov's PhD thesis. The central theme is the exploration of the extent to which good model theoretic properties (such as weak o-minimality, its strengthening *quite* o-minimality, countable categoricity, model completeness) are preserved when a structure is expanded by predicates with certain properties. Such preservation results belong to a very prominent theme in modern model theory. For example, there are many beautiful recent papers examining the case when the predicate is unary and is interpreted by an elementary substructure (leading to 'lovely pairs'), or an independent set (leading to 'H-structures') or a 'generic' set. A related theme, leading to the 'Shelah expansion', is when a structure M is expanded by 'externally definable sets', that is, traces in M of definable sets in an elementary extension of M.

The first four chapters of the thesis consist of basic definitions (Chapter 1), a glossary of notation (Chapter 2), a summary of the main results (Chapter 3), and an extended and interesting historical review of the wider context (Chapter 4). The latter shows that Sayan Baizhanov has a broad understanding of pure model theory and the wider setting of his work. The original results are entirely in Chapters 5-8.

By a 2001 result of B.S. Baizhanov, weak o-minimality is preserved by any expansion by a convex predicate (that is, predicate interpreted by a convex set). B. Kulpeshov has conducted a deep analysis of countably categorical weakly o-minimal theories of finite *convexity rank*. In Chapter 5, these ideas are put together.

Sayan Baizhanov shows that if M is a countably categorical weakly o-minimal structure of convexity rank k, then any expansion of M by finitely many convex predicates preserves these properties. He also proves essentially the same result but with 'weak o-minimality' replaced by the stronger condition 'quite o-minimality'.

The focus of Chapter 6 is on expansions of countably categorical weakly o-minimal structures by predicates interpreted by equivalence relations with convex classes. In the case when M is 1-indiscernible (that is, has transitive automorphism group) and has convexity rank 1, a necessary and sufficient condition is given for an expansion by such an equivalence relation to be countably categorical weakly o-minimal. This result is then extended to the case when M has finite convexity rank, and the expansion is by finitely many equivalence relations with convex classes.

In Chapter 7, expansions by more general binary predicates (but still satisfying extra properties) are considered. The base structure M is assumed to be countably categorical weakly o-minimal of convexity rank 1, and necessary and sufficient conditions are given (initially in the 1-indiscernible case) for such an expansion to itself be countably categorical.

Chapter 8 concerns the notions of *external definability* and *Shelah expansion* mentioned above. Countable categoricity is no longer an assumption in this chapter. Sayan Baizhanov shows first that if M has weakly o-minimal model complete theory, then an expansion by a convex predicate with a certain additional technical condition (with respect to a 'quasisolitary' type) preserves model completeness. He then summarises a number of results and approaches around the Shelah expansion (Macpherson-Marker-Steinhorn, B.S. Baizhanov, Shelah, Verbovskiy). He concludes with a result showing that an expansion of a weakly o-minimal ordered group by a certain kind of binary predicate is again weakly o-minimal.

Summary. The body of work in this thesis is new, interesting, and is I believe correct. Many intermediate results not mentioned above are also proved – for example technical quantifier elimination results necessary for the main theorems. The thesis is also rich in enlightening examples. I understand much of it has already been published and disseminated in talks, and it does indeed look suitable for publication in good international journals. It belongs to a research area of wide current interest, with many ramifications. The draft I have seen is not final, and there is still room for substantial improvement in the presentation (and I have made suggestions to Sayan Baizhanov).

I have had interesting discussions with Sayan Baizhanov on his two visits to Leeds, and my two visits to Kazakhstan (to Almaty, and to a conference in Astana). I have observed that over this period there has been a considerable increase in the sophistication and depth of his understanding of model theory. I am satisfied that the thesis meets all the requirements for the degree of Doctor of Philosophy (PhD), and that Sayan Baizhanov deserves the degree of Doctor of Philosophy (PhD) in Mathematics.

Your sincerely,

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